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Recurrence and transience properties of multi-dimensional diffusion processes in selfsimilar and semi-selfsimilar random environments

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1 Introduction

This note is a short review of the papers [8] and [9].

It is well-known that a multi-dimensional standard Brownian motion, which consists of d independent one-dimensional standard Brownian motions, is recurrent if $d=1$ or 2, and transient otherwise. We consider limiting behaviors of multi-dimensional diffusion processes in selfsimilar and semi-selfsimilar random environments.

Let $\mathcal{W}$ be the space of $\mathbb{R}$-valued functions W satisfying the following:

(i) $W(0)=0$,

(ii) W is right continuous and has left limits on $[0, \infty)$,

(iii) W is left continuous and has right limits on $(-\infty, 0]$.

Following [18], we set a probability measure Q on $\mathcal{W}$ such that $\{W(x), x\geq 0, Q\}$ and $\{W(-x), x\geq 0, Q\}$ are independent strictly semi-stable Lévy processes with index α,

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which have the following semi-selfsimilarity:

$$\{W(x), x \in \mathbb{R}\} \stackrel{d}{=} \{a^{-1/\alpha}W(ax), x \in \mathbb{R}\} \quad \text{for some } a > 0, \tag{1.1}$$

where $\stackrel{d}{=}$ denotes the equality in all joint distributions. This a is called an epoch. We set

$$r = \inf\{a > 1 : a \text{ satisfies (1.1)}\}. \tag{1.2}$$

In this paper, we call (W, Q) an (r, α)-semi-stable Lévy environment. If $r = 1$, (W, Q) is not only semi-selfsimilar but selfsimilar. In this case, we call (W, Q) an α-stable Lévy environment. Refer [11] to more properties of semi-stable Lévy processes.

For a fixed W, we consider a d-dimensional diffusion process starting at 0, $X_W = \{X_W^k(t), t \geq 0, k = 1, 2, 3, \ldots, d\}$ whose generator is

$$\sum_{k=1}^{d} \frac{1}{2} \exp\{W(x_k)\} \frac{\partial}{\partial x_k} \left\{ \exp\{-W(x_k)\} \frac{\partial}{\partial x_k} \right\}. \tag{1.3}$$

We regard values of W at different d points as a multi-parameter environment. Such X_W is constructed by d independent standard Brownian motions with a scale transformation and a time change (c.f. [6]). Each component of X_W is symbolically described by

$$dX_W^k(t) = dB^k(t) - \frac{1}{2} W'(X_W^k(t))dt, \quad X_W^k(0) = 0, \quad \text{for } k = 1, 2, 3, \ldots, d,$$

where $B^k(t)$ is a one-dimensional standard Brownian motion independent of the environment (W, Q).

In the case where $d = 1$ and (W, Q) is a Brownian environment, Brox showed that the distribution of $(\log t)^{-2} X_W(t)$ converges weakly as $t \to \infty$ in [1]. This shows that X_W moves very slowly by the effect of the environment. This diffusion process is a continuous model of random walks in random environments studied by Solomon [13] and Sinai [12], and X_W is often called a Brox-type diffusion. Following Brox's result, Tanaka studied the cases of α-stable Lévy environments and showed the convergence theorem with the scaling $(\log t)^{-\alpha} X_W(t)$ under the assumption that $Q\{W(1) > 0\} > 0$ in [18]. Tanaka's results were extended to the cases of (r, α)-semi-stable Lévy environments in [15].

In view of the subdiffusive property of the Brox-type diffusion, we expect to see an exotic limiting behavior of multi-dimensional Brox-type diffusions. We give a brief review of investigations related to multi-dimensional Brox-type diffusions. Fukushima *et al.* showed the recurrence of the diffusion process whose generator is

$$\frac{1}{2}e^{W(|\mathbf{x}|)}\sum_{k=1}^{d}\frac{\partial}{\partial x_k}\left\{e^{-W(|\mathbf{x}|)}\frac{\partial}{\partial x_k}\right\},$$

where $|\mathbf{x}| = \sqrt{x_1^2+x_2^2+x_3^2+\cdots+x_d^2}$ and W is a one-dimensional standard Brownian motion in [2]. In the case where the environment is Lévy's Brownian motion $W(\mathbf{x})$ with a multi-dimensional time, Tanaka showed the recurrence of the diffusion process for almost all environments in any dimension in [19]. These results are shown by Ichihara's recurrent test introduced in [5]. Mathieu studied asymptotic behaviors of multi-dimensional diffusion processes in random environments by using Dirichlet form and showed the convergence theorem in the case where the environment is a non-negative reflected Lévy's Brownian motion in [10]. Following the study, Kim obtained some limit theorems of the multi-dimensional diffusion processes in [7]. He showed the convergence theorem in the case where the random environment consists of d independent one-dimensional reflected non-negative Brownian environments, which is a model studied in [16]. In [17], the multi-dimensional diffusion process consisting of d independent Brox-type diffusions was studied and the recurrence of the process for almost all environments in any dimension was shown. Recently, Gantert *et al.* showed the recurrence of d independent random walks in random environments, which corresponds to a model studied in [17], by using estimates of quenched return probabilities to the origin of the one-dimensional random walks in random environments in [4].

2 Selfsimilar and semi-selfsimilar Lévy random environments' case

Following the previous studies, we consider limiting behaviors of diffusion processes in (r, α)-semi-stable Lévy environments as (1.1) and (1.2), which are extensions of models studied in [4] and [17]. We call $\{W(x), x \geq 0, Q\}$ a subordinator if it is an increasing (r, α)-semi-stable or α-stable Lévy environment. We obtain some conditions of the random environments which imply the dichotomy of recurrence and transience of d-dimensional diffusion processes corresponding to the generator (1.3) as follows:

Theorem 1. (I) If $\{-W(x), Q\}$ is not a subordinator, then X_W is recurrent for almost all environments in any dimension.

(II) If $\{-W(x), Q\}$ is a subordinator, then X_W is transient for almost all environments in any dimension.

We next consider d-dimensional diffusion processes consisting of d independent Brox-type diffusions. Let Q_k be the probability measure on $\mathcal{W}$ such that

(i) $\{W_k(-x_k), x_k \geq 0, Q_k\}$ is an (l_k, α_k)-semi-stable or an α_k-stable Lévy environment,

(ii) $\{W_k(x_k), x_k \geq 0, Q_k\}$ is an (r_k, β_k)-semi-stable or a β_k-stable Lévy environment,

(iii) they are independent.

We define an environment $(\mathbf{W}, \mathbf{Q})$ by $\{(W_k, Q_k), k = 1, 2, 3, \ldots, d\}$ with independent (W_k, Q_k)'s. We remark that Suzuki studied the one-dimensional case with independent an α-stable and a β-stable Lévy environment, and obtained some convergence theorems in [14]. We also call $\{W_k(-x_k), x_k \geq 0, Q_k\}$ a subordinator if it is a decreasing (l_k, α_k)-semi-stable or α_k-stable Lévy environment. For a fixed $\mathbf{W}$, we consider a d-dimensional diffusion process starting at 0, $X_{\mathbf{W}} = \{X^{(k)}_{W_k}(t), t \geq 0, k = 1, 2, 3, \ldots, d\}$ whose generator is

$$\sum_{k=1}^{d} \frac{1}{2} \exp\{W_k(x_k)\} \frac{\partial}{\partial x_k} \left\{ \exp\{-W_k(x_k)\} \frac{\partial}{\partial x_k} \right\}. \tag{2.1}$$

On the d-dimensional diffusion processes, we obtain the following dichotomy theorem:

Theorem 2. (I) If neither $\{-W_k(-x_k), x_k \geq 0, Q_k\}$ nor $\{-W_k(x_k), x_k \geq 0, Q_k\}$ is a subordinator for any k, then $X_{\mathbf{W}}$ is recurrent for almost all environments in any dimension.

(II) If either $\{-W_k(-x_k), x_k \geq 0, Q_k\}$ or $\{-W_k(x_k), x_k \geq 0, Q_k\}$ is a subordinator for some k, then $X_{\mathbf{W}}$ is transient for almost all environments in any dimension.

3 Multi-dimensional Gaussian environments

In this section, we consider the recurrence of the diffusion process X_W given by the following generator:

$$\frac{1}{2}(\Delta - \nabla W \cdot \nabla) = \frac{1}{2}e^W \sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left\{ e^{-W} \frac{\partial}{\partial x_k} \right\}, \tag{3.1}$$

where W is a Gaussian field on $\mathbb{R}^d$ i.e., $\{W(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d\}$ is a family of random variables such that the $\mathbb{R}^d$-valued random variable $(W(\mathbf{x}_1), W(\mathbf{x}_2), \ldots, W(\mathbf{x}_n))$ has an n-dimensional Gaussian distribution for all $n \in \mathbb{N}$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^d$. We assume that W is continuous on $\mathbb{R}^d$ almost surely, $W(\mathbf{0}) = 0$, and that $E[W(\mathbf{x})] = 0$ for $\mathbf{x} \in \mathbb{R}^d$. We can construct the diffusion process X_W associated with the generator above by a random time-change of the diffusion process associated with the Dirichlet form:

$$\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}^d} (\nabla f \cdot \nabla g)\, e^{-W} dx.$$

Hence, the existence of the diffusion process X_W associated with (3.1) is guaranteed (see [3]). Let $K(\mathbf{x}, \mathbf{y}) := E[W(\mathbf{x})W(\mathbf{y})]$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. Fixing $r > 1$ we denote the set $\{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| < r^n\}$ by E_n for $n \in \mathbb{N}$. We also denote $E_n \setminus E_{n-1}$ by D_n. Fixing $H > 0$, we define a mapping T from Borel measurable functions on $\mathbb{R}^d$ to themselves by

$$Tf(\mathbf{x}) := r^{-H} f(r\mathbf{x}), \tag{3.2}$$

and let $T_n := T^n$ for $n \in \mathbb{N}$. Now we assume that the law of TW equals to that of W. Then, T is a measure preserving transformation. For the Gaussian field W, we obtain the following results:

Theorem 3. Let W be a Gaussian field on $\mathbb{R}^d$ satisfying that

(i) there exists a positive constant ε such that

$$\inf_{x \in D_1} \int_{D_1} K(\mathbf{x}, \mathbf{y}) dy \geq \varepsilon,$$

(ii) the law of $T_n W$ equals to that of W for all $n \in \mathbb{N}$ and that

$$\lim_{n \to \infty} r^{-nH} \sup_{\mathbf{x}, \mathbf{y} \in D_1} K(r^n \mathbf{x}, \mathbf{y}) = 0.$$

Then, the diffusion process X_W associated with the generator (3.1) is recurrent for almost all environments W.

In the case where environments are fractional Brownian fields on $\mathbb{R}^d$, we can apply Theorem 3 and show the recurrence of the diffusion process X_W given by the generator (3.1). For a given $H \in (0,1)$, let W be a Gaussian random environment which satisfying that $W(\mathbf{0}) = 0$, $E[W(\mathbf{x})] = 0$ for $\mathbf{x} \in \mathbb{R}^d$, and that the covariance between $W(\mathbf{x})$ and $W(\mathbf{y})$ is given by

$$K(\mathbf{x},\mathbf{y}) := \frac{1}{2}\left(|\mathbf{x}|^{2H} + |\mathbf{y}|^{2H} - |\mathbf{x}-\mathbf{y}|^{2H}\right), \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^d.$$

Note that the law of Gaussian random environments are determined by the means and the covariance. The random field W is called a fractional Brownian field. When $H = 1/2$, it is called Lévy's Brownian motion (c.f. [19]). It is easy to see that the environment W is a selfsimilar random environment with the mapping (3.2). The parameter H is called the Hurst parameter. Now we can show the following theorem as an application of Theorem 3.

Theorem 4. Let W be a fractional Brownian field on $\mathbb{R}^d$ with the Hurst parameter $H \in (0,1)$. Then, the process X_W given by the generator (3.1) is recurrent for almost all environments W.

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